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Isometry of AdS_2 and the $c = 1$ matrix model

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Abstract: Implications of the SL$(2,\mathbb{R})$ symmetry of the $c = 1$ matrix models are explored. Based on the work of de Alfaro, Fubini and Furlan, we note that when the Fermi sea is drained, the matrix model for 2 dimensional string theory in the linear dilaton background is equivalent to the matrix model of AdS_2 recently proposed by Strominger, for which SL$(2,\mathbb{R})$ is an isometry. Utilizing its Lie algebra, we find that a topological property of AdS_2 is responsible for quantizing D0-brane charges in type 0A theory. We also show that the matrix model faithfully reflects the relation between the Poincare patch and global coordinates of AdS_2.

Keywords: Field Theories in Lower Dimensions, M(atrix) Theories, String Field Theory, Global Symmetries.

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Contents

1. Introduction

Motivated by the problems with unstable D-branes, the $c = 1$ matrix model [1] (for reviews see [2]–[4]) has recently attracted a lot of attention [5] as the simplest string theory from which we might learn something useful.

The $c = 1$ matrix model is equivalent to a theory of free fermions Ψ in 1+1 dimensional spacetime with the hamiltonian [1]

$$H = -\frac{1}{2}\partial_x^2 + V(x)\,, \qquad V(x) = -\frac{x^2}{2}\,. \tag{1.1}$$

It is dual to 2 dimensional bosonic string theory or type 0B theory [6, 7] depending on whether we fill one side or both sides of the potential. By filling the Fermi sea differently and an orbifolding, the same model was also conjectured to be dual to type IIB theory [8].

[1] More precisely we should consider the field theory, and the hamiltonian is $\int dx \Psi^\dagger H \Psi$, where $\Psi(x)$ is the 2nd quantized fermion field. We will use the notation of quantum mechanics although everything can be extended to the 2nd quantized theory.

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A slight deformation of the potential

$$V(x) = -\frac{x^2}{2} + \frac{M}{2x^2}\,. \tag{1.2}$$

was conjectured [9] to lead to bosonic string theory in 2 dimensions in the background of a black hole of mass M. More recently it was also conjectured [6, 7] to be dual to type 0A string theory in the linear dilaton background with RR electric field proportional to q, where

$$M = q^2 - \frac{1}{4}\,. \tag{1.3}$$

In the limit $M \to \infty$ (or $x \to 0$), we can scale x and ignore the quadratic term in the potential. Hence

$$V(x) = \frac{M}{2x^2}\,. \tag{1.4}$$

This model was conjectured [10] to be dual to type 0A string theory in the AdS_2 background. The isometry group $\mathrm{SL}(2,\mathbb{R})$ of AdS_2 is identified with a symmetry algebra of this matrix model. The generators are

$$H = \frac{1}{2}\left(p^2 + \frac{M}{x^2}\right), \qquad K = \frac{1}{2}x^2\,, \qquad D = -\frac{1}{4}\left(xp + px\right), \tag{1.5}$$

where $p = -i\partial_x$ is the conjugate momentum of x. They satisfy the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra

$$[H, D] = iH\,, \qquad [K, D] = -iK\,, \qquad [H, K] = 2iD\,. \tag{1.6}$$

The ground states of matrix models are specified by a single parameter μ which is the energy at the surface of the Fermi sea. It is dual to the amplitude of a static tachyon background. In the AdS_2 model the potential (1.4) is bounded from below at zero, so $\mu \geq 0$. The ground states with $\mu > 0$ spontaneously break the $\mathrm{SL}(2,\mathbb{R})$ symmetry. Only the unique state with no fermion ($\mu = 0$) preserves the isometry, and is matched to the invariant vacuum of AdS_2.

One might be puzzled by the fact that the $\mathrm{SL}(2,\mathbb{R})$ symmetry exists also for other matrix models as part of the W_∞ algebra. But of course $\mathrm{SL}(2,\mathbb{R})$ can not be the isometry of the dual target spaces which are asymptotically flat. The resolution is that the $\mathrm{SL}(2,\mathbb{R})$ symmetry can be understood as the isometry of the matrix model in a way we will explain later, and the isometry is spontaneously broken by the ground state chosen for the other matrix models. Nevertheless, we will see that the $\mathrm{SL}(2,\mathbb{R})$ symmetry has important implications also for other matrix models.

Utilizing the $\mathrm{SL}(2,\mathbb{R})$ generators, the authors of [11] showed that all quantum theories with hamiltonians of the form

$$H = \frac{p^2}{2} + V(x)\,, \qquad V(x) = \frac{a}{2}x^2 + \frac{M}{2x^2} \tag{1.7}$$

with the same parameter M, but arbitrary coefficient a, are related to one another by coordinate transformations. This has many striking implications. For $M = 0$, the undeformed matrix model ($a = -1$) can be identified with the simple harmonic oscillator ($a = 1$), and also with the theory with a trivial potential ($a = 0$), by simple changes of coordinates. For $M > 0$, the deformed matrix model ($a = -1$) can be matched with the type 0A theory in AdS_2 ($a = 0$). Yet these theories have completely different asymptotic behaviors! As we will show below, this is a reflection of different coordinate patches of AdS_2 which are not isomorphic.

As the potentials in (1.2) and (1.4) can be viewed as the same theory written in two sets of coordinates, the $\mu \to -\infty$ limit of type 0A matrix model can be identified with the vacuum of the AdS_2 matrix model. In other words, type 0A theory in AdS_2 background should be identified with the result of tachyon condensation from the linear dilaton background.[2] Due to the similarity between type 0A and 0B, we conjecture that the same is true for type 0B theory as well.

The coordinate transformations relating theories with different a in (1.7) are not always bijective. Some coordinate systems only cover a small part of the spacetime defined by another set of coordinates. We will see that properties of coordinate transformations of the matrix model reflect those of the target space in string theory. It was proposed [10] that the hamiltonian (1.7) with $a = 1$ (i.e., $H + K$) is dual to AdS_2 in global coordinates, while (1.4) is dual to the Poincare patch, which only covers half of the former. Based on this proposal, we will show that the RR flux background q needs to be quantized in order for the global time of AdS_2 to be compactified.

In addition, we will show that, by adding a suitable (time-dependent) Fermi surface, we can use the hamiltonian (1.7) with any a to describe string theory in the linear dilaton background. As the asymptotic behavior of particles for $a > 0, a = 0$ and $a < 0$ are drastically different, the phenomenology of each model appears to be very different, although the encoded information is (almost) equivalent. This reflects the observer dependence familiar in the context of general relativity. By analogy with AdS_2, there should exist a coordinate transformation (possibly combined with field redefinitions) of string theory/supergravity in the linear dilaton background which extends the spacetime beyond the region manifest in the old description.

In the appendix A we prove that (1.7) exhausts all possibilities of nontrivial isometries for 1 dimensional quantum mechanics with the standard kinetic term, and a time-independent potential. The proof that all theories defined by (1.7) with the same M are equivalent at the quantum level are given in appendix B.

2. SL(2, $\mathbb{R}$) symmetry

The quantum mechanics defined by (1.4) was extensively studied [11] as an example of quantum mechanics with conformal symmetry. Assuming that the kinetic energy is standard $\frac{1}{2}p^2$, by simple dimensional analysis one can see that (1.4) is the only potential consistent with conformal symmetry. (Conformal quantum mechanics with more than one variables and general kinetic terms were discussed in [12].) The SL(2, $\mathbb{R}$) symmetry (1.5) was found in [11] as the conformal symmetry.

[2] As the closed string tachyons are massless in 2 dimensions, tachyon condensation might never happen. What we mean here is to tune the parameter μ of the tachyon field by hand to $-\infty$.

In this section we will first review how the $\mathrm{SL}(2,\mathbb{R})$ symmetry is a symmetry of conformal transformations of the time variable together with a time-dependent scaling of the spatial coordinate for the matrix model [11]. This could come as a bit of surprise for those who are not familiar with the results of [11], since the spacetime coordinates in the matrix model are nonlocally related to the coordinates of target space. (The nonlocal transformation are determined by the leg factors [4, 9].)

Then we will use the Lie algebra generators, which are dual to Killing vectors in AdS_2, to define new time coordinates, and find that the new hamiltonians are all of the form (1.7) with the same M. Furthermore, we will give a precise matching between coordinate systems for the matrix model and those of the AdS_2 space.

2.1 Isometry of quantum mechanics

An infinitesimal general coordinate transformation is generated by a differential operator of the form

$$\mathcal{D} = iA\partial_t + iB\partial_x + C\,, \tag{2.1}$$

where A, B, C are functions of x and t. Classically C can be dropped, and the infinitesimal coordinate transformation is

$$\delta t = \epsilon A\,, \qquad \delta x = \epsilon B\,. \tag{2.2}$$

The final result will justify why C should be allowed for quantum mechanics. For a Schrödinger equation

$$((\mathrm{EOM})) = i\partial_t - H\,, \tag{2.3}$$

$\mathcal{D}$ generates a symmetry if

$$(\mathrm{EOM})\mathcal{D} = \mathcal{D}'(\mathrm{EOM}) \tag{2.4}$$

for some well defined operator $\mathcal{D}'$ so that a solution of the Schrödinger equation is still a solution after the transformation defined by $\mathcal{D}$, i.e.,

$$(\mathrm{EOM})\psi = 0 \quad \Rightarrow \quad (\mathrm{EOM})e^{i\epsilon\mathcal{D}}\psi = 0\,. \tag{2.5}$$

Note that $\mathcal{D}'$ does not have to equal $\mathcal{D}$. The condition (2.4) is equivalent to

$$[(\mathrm{EOM}), \hat{\mathcal{D}}] = 0\,, \qquad \hat{\mathcal{D}} = AH - Bp + C\,, \tag{2.6}$$

where $\hat{\mathcal{D}}$ is $\mathcal{D}$ with $i\partial_t$ replaced by the hamiltonian H, and $i\partial_x$ by $-p$. ($\mathcal{D}$ and $\hat{\mathcal{D}}$ are equivalent when acting on solutions of the Schrödinger equation.) The generators (1.5) are to be identified with $\hat{\mathcal{D}}$'s at $t = 0$ for the matrix model.

We will refer to the coordinate transformations which preserve the Schrödinger equation as the quantum mechanics isometry. It is natural to ask when a quantum theory admits nontrivial isometry. Any time-independent hamiltonian is an isometry generator of time translation. We prove in appendix A for the standard kinetic energy $\frac{1}{2}p^2$ that the only time-independent hamiltonian with more than one isometry generators is of the form (1.7). There are 3 isometry generators H, K, D (1.5) for a generic M. If $M = 0$, there are 2 additional generators, which can be identified with the creation and annihilation operators of the simple harmonic oscillator when $a > 0$.

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In terms of the coordinates in the AdS_2 matrix model (1.4), the SL(2, $\mathbb{R}$) isometry is represented as the projective transformation of time and a scaling of space [11]

$$t \to \frac{\alpha t+\beta}{\gamma t+\delta}, \qquad x \to \frac{1}{\gamma t+\delta} x . \tag{2.7}$$

2.2 Killing operators and coordinate transformations

In analogy with riemannian geometry, the symmetry generators preserving Schrödinger equations are reminiscent of Killing vectors, which preserve the metric, and it is natural to use them to define new time coordinates. Of course, one can always perform arbitrary general coordinate transformations or change of variables to rewrite a theory. The special features of the time coordinates chosen by Killing operators are that their conjugate hamiltonians are time-independent and of the standard form $\frac{1}{2}p^2 + V(x)$.

Take a generic SL(2, $\mathbb{R}$) generator

$$G = \alpha H + \beta D + \gamma K , \tag{2.8}$$

and we would like to introduce a new coordinate τ such that the old and new Schrödinger equations are equivalent

$$i\frac{\partial}{\partial t}\Psi = H\Psi \Longleftrightarrow i\frac{\partial}{\partial \tau}\Psi' = G\Psi' , \tag{2.9}$$

where Φ' is related to Φ through a unitary transformation and change of measure. (See appendix B.) It turns out that, remarkably, τ is simply a function of t [11]

$$d\tau = \frac{dt}{\alpha+\beta t+\gamma t^2} . \tag{2.10}$$

For the range of t satisfying

$$f(t) = (\alpha+\beta t+\gamma t^2) > 0 , \tag{2.11}$$

τ is a legitimate reparametrization of time. Furthermore, after a simultaneous scaling of the spatial coordinate

$$\sigma = \frac{x}{(\alpha+\beta t+\gamma t^2)^{1/2}} , \tag{2.12}$$

G is again of the form (1.7) [11]

$$G = \frac{p_\sigma^2}{2} + V_G , \qquad V_G = \frac{\Delta}{8}\sigma^2 + \frac{M}{2\sigma^2} , \tag{2.13}$$

where

$$\Delta = 4\alpha\gamma - \beta^2 . \tag{2.14}$$

The new coordinate system is as good as the old one as long as (2.11) is satisfied.

Here we verify the claim above in the classical theory. The action for the potential (1.4) is

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{M}{2x^2}\right) . \tag{2.15}$$

From (2.10) and (2.12), it follows that

$$\frac{dx}{dt} = f^{-1/2}(\partial_\tau \sigma + \frac{1}{2}(\partial_t f)\sigma)\,. \tag{2.16}$$

Plugging it in the action and integrating by parts, we find

$$S = \left[\frac{1}{2}(\partial_t f)x^2\right]_{\tau_0}^{\tau_1} + \int d\tau \left(\frac{1}{2}(\partial_\tau\sigma)^2 - \frac{\Delta}{8}\sigma^2 - \frac{M}{2\sigma^2}\right). \tag{2.17}$$

This is exactly the action for the hamiltonian G (2.13). Due to the boundary term in (2.17), a unitary transformation is involved in the quantum version.

At the quantum level, the derivation is given in appendix B. We will show there that

$$\Phi' = f^{1/4} U^\dagger \Phi\,, \tag{2.18}$$

where U is a phase factor (B.8). The factor $f^{1/4}$ comes from the change of measure due to the time-dependent scaling of the spatial coordinate. It is there so that the inner products for Φ' and Φ agree

$$\int dx \Phi^\dagger \Phi = \int d\sigma \Phi'^\dagger \Phi'\,. \tag{2.19}$$

It is a very intriguing fact that all hamiltonians of the form (1.7) are different descriptions of the same theory related by simple coordinate transformations (2.10) and (2.12).

2.3 Coordinate patches: $S_- \subset S_0 \subset S_+$

A redefinition (scaling) of the coordinates ($\sigma \to \lambda\sigma$, $\tau \to \lambda^2\tau$) has the effect of scaling the coefficient of the σ^2 term by λ^4. We have 3 classes of G that can not be related to each other via such a scaling. They are examplified by ($\alpha = 1, \beta = 0, \gamma = 1$), ($\alpha = 1, \beta = 0, \gamma = 0$) and ($\alpha = 1, \beta = 0, \gamma = -1$). For notation, we will equip variables with subscripts $+, 0, -$, according to the values of γ, and refer to the corresponding matrix models by S_+, S_0, and S_-, respectively. The Killing operators associated to S_+, S_0, S_- are $(H + K), H, (H - K)$. According to (2.13), their potentials are

$$S_+ : \; V_+ = \frac{1}{2}x_+^2 + \frac{M}{2x_+^2}\,, \qquad S_0 : \; V_0 = \frac{M}{2x_0^2}\,, \qquad S_- : \; V_- = -\frac{1}{2}x_-^2 + \frac{M}{2x_-^2}\,. \tag{2.20}$$

The coordinate transformations are given by (2.10) and (2.12)

$$t_0 = \tan(t_+) = \tanh(t_-)\,, \qquad x_0 = \sec(t_+)x_+ = \mathrm{sech}(t_-)x_-\,. \tag{2.21}$$

It follows that the momenta are related by

$$p_0 = \cos(t_+)p_+ + \sin(t_+)x_+ = \cosh(t_-)p_- - \sinh(t_-)x_-\,. \tag{2.22}$$

From these expressions we can see that the origins of all time coordinates coincide. At $t = 0$, the spatial coordinates and conjugate momenta also coincide.

For later use, we list here the solutions of the classical equations of motion

$$x_+ = \sqrt{A_+^2 \cos(2(t_+ - T_+)) + \sqrt{A_+^4 + M}}\,, \tag{2.23}$$

$$x_0 = \sqrt{2E_0(t_0 - T_0)^2 + \frac{M}{2E_0}}\,, \tag{2.24}$$

$$x_- = \sqrt{A_-^2 \cosh(2(t_- - T_-)) \pm \sqrt{A_-^4 - M}}\,, \tag{2.25}$$

where we assumed that $M > 0$, and that the particle stays on the positive side of the real line. The parameters are related by

$$\begin{aligned} A_+^2 &= \sqrt{\left(E_0(T_0^2 - 1) + \frac{M}{4E_0}\right)^2 + 4E_0^2 T_0^2}\,, \qquad T_+ = -\tan^{-1}\left(\frac{2E_0 T_0}{E_0(T_0^2 - 1) + \frac{M}{4E_0}}\right), \\ A_-^2 &= \sqrt{\left(E_0(T_0^2 + 1) + \frac{M}{4E_0}\right)^2 - 4E_0^2 T_0^2}\,, \qquad T_- = -\tanh^{-1}\left(\frac{2E_0 T_0}{E_0(T_0^2 + 1) + \frac{M}{4E_0}}\right). \end{aligned} \tag{2.26}$$

For $M = 0$, the solutions are

$$x_+ = A(\sin(t_+) - T_0 \cos(t_+))\,, \tag{2.27}$$

$$x_0 = A(t_0 - T_0)\,, \tag{2.28}$$

$$x_- = A(\sinh(t_-) - T_0 \cosh(t_-))\,. \tag{2.29}$$

These solutions are related to each other through (2.21). But they are not the $M \to 0$ limit of (2.23)–(2.25), since the $M = 0$ theory is not continuously connected to theories with $M > 0$.

The coordinate transformations (2.21) are not 1-1 mappings. Roughly speaking, the coordinate systems of S_-, S_0, S_+ form a cascade of patches

$$S_- \subset S_0 \subset S_+\,. \tag{2.30}$$

The whole range $\mathbb{R}$ of time for a smaller patch is mapped to a finite interval in a larger patch. More precisely,

$$(-\infty < t_- < \infty) \to (-1 < t_0 < 1) \to \left(-\frac{\pi}{4} < t_+ < \frac{\pi}{4}\right), \tag{2.31}$$

$$(-\infty < t_0 < \infty) \to \left(-\frac{\pi}{2} < t_+ < \frac{\pi}{2}\right). \tag{2.32}$$

One can check that the condition (2.11) is satisfied and t_0, t_+, and t_- are legitimate time coordinates within the ranges shown above.

So far we have treated t_+ as a parameter of $\mathbb{R}$. Later we will see that t_+ should be compactified on a unit circle. But the hierarchical relation (2.30) remains the same.

2.4 Matrix model isometry vs. spacetime isometry

Each of these coordinate systems in the matrix model is matched to a coordinate system in the dual theory of AdS_2 according to the associated Killing operators. The Killing vectors

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of AdS_2 are

$$H = i\partial_t\,, \qquad D = i\left(t\partial_t + \sigma\partial_\sigma\right), \qquad K = i\left((t^2+\sigma^2)\partial_t + 2t\sigma\partial_s\right), \tag{2.33}$$

where we used the same notation to identify $\mathrm{SL}(2,\mathbb{R})$ generators in AdS_2 and the matrix model. Here (t,σ) are the coordinates of the Poincare patch. Its dual matrix model description is S_+.

The generator $(H+K)$ can be written as $i\partial_\tau$ where τ is the time in the global coordinates (τ,ω)

$$\tau \pm \omega = 2\tan^{-1}(t\pm\sigma)\,. \tag{2.34}$$

The dual is S_0. Similarly, defining another set of coordinates

$$\tau' \pm \omega' = 2\tanh^{-1}(t\pm\sigma)\,, \tag{2.35}$$

one finds $H-K = i\partial_{\tau'}$. This Killing vector $H-K$ is time-like only in a small subset of the Poincare patch of AdS_2. Apparently it is dual to S_-. While the asymptotic behaviors of a free fermion in the matrix model in different coordinate systems S_+, S_0, S_- are qualitatively distinctive, the asymptotic solutions of a free field in AdS_2 are equally different in the dual coordinate systems.

We note that the relation (2.30) is mimicked by their AdS_2 cousins. Despite the duality, this is nontrivial because the duality map is nonlocal.

3. Target space geometry

In this section we first review the motivation for proposing the matrix model (1.4) for AdS_2 space. Then we comment on the connections among different matrix models. Finally we will use the results of the previous section to show that the quantization of the RR flux in type 0A string theory can be connected with the topology of AdS_2 through the matrix models.

3.1 AdS_2 as near horizon geometry

The supergravity solution of type 0A theory with background RR electric field proportional to q is [13]

$$ds^2 = \left(1+\frac{q^2}{8}(\Phi-\Phi_0-\frac{1}{2})e^{2\Phi}\right)(-dt^2+d\sigma^2)\,, \tag{3.1}$$

$$\sigma(\Phi) = \frac{1}{\sqrt{2}}\int^{\Phi}\frac{d\Phi'}{1+\frac{q^2}{8}(\Phi'-\Phi_0-\frac{1}{2})e^{2\Phi'}}\,, \tag{3.2}$$

$$\Phi_0 = -\log\frac{q}{4}\,, \tag{3.3}$$

which is asymptotically flat in the $\sigma\to\infty$ limit. If we scale the spatial coordinate around the "near horizon" region $\sigma\to\infty$, we arrive at the AdS_2 geometry

$$ds^2 = \frac{1}{4\sigma^2}(-dt^2+d\sigma^2)\,, \qquad \Phi=\Phi_0=-\log\frac{q}{4}\,. \tag{3.4}$$

Correspondingly, the deformed matrix quantum mechanics dual to type 0A theory with RR flux q is defined by (1.2). In terms of the generators defined in (1.5), the hamiltonian is $H - K$. If we scale x, H and K are scaled accordingly

$$x \to \lambda x\,, \qquad H \to \lambda^{-2} H\,, \qquad K \to \lambda^{2} K\,. \tag{3.5}$$

Thus, the hamiltonian approaches to H as we zoom in x. This is simply saying that if we zoom in around the region very close to $x = 0$, we can eventually ignore the x^2 term in the potential. Similarly, any Fermi surface with $\mu \leq 0$ will eventually be out of sight as we zoom in around $x = 0$. Hence it is natural to conjecture [10] that (1.4) defines the matrix model for AdS_2, and that the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum of the AdS_2 background corresponds to the vacuum without fermion in the matrix model.

3.2 Connections among matrix models

According to section 2, the AdS_2 matrix model (1.4), which is of the form of S_0, can also be written as S_+ or S_-. The distinctive feature of the AdS_2 theory is that the ground state has no fermion. When the Fermi sea is filled to a finite energy μ in S_-, the matrix model is dual to string theory in the linear dilaton background with a tachyon field proportional to μ. This means that the back-reaction of the tachyon field changes the target space geometry from an asymptotically flat space to AdS_2 in the limit of tachyon condensation $\mu \to -\infty$. By tunning the tachyon amplitude, we can interpolate between Minskowski space ($\mu = 0$) and AdS_2 ($\mu = -\infty$). This gives an explicit example of how large fluctuations of the Fermi sea correspond to changes of the background geometry.

For bosonic or type 0B theory, there is no analogous scaling argument on the supergravity side for AdS_2 as in section 3.1. However, for the undeformed matrix quantum mechanics, the vacuum with no fermion is still $\mathrm{SL}(2,\mathbb{R})$ invariant. Based on the similarity among type 0A, 0B and bosonic theories, we propose that AdS_2 is also the spacetime geometry for type 0B theory in the limit $\mu \to -\infty$.

If the correspondence between S_+ and the global AdS_2 theory is correct, we would expect that a similar correspondence should persist when a Fermi sea is introduced. That is, there may exist an alternative description of 2 dimensional string theory with the linear dilaton background, which contains the old story as a partial description of the full theory. Since complicated field redefinition is involved in matching the matrix model with the spacetime physics, it is possible that the incompleteness of the usual description is not simply referring to the spacetime, but to the set of observables in string theory. Currently we do not have a candidate for this theory, we only know that it should be dual to S_+ with a time-dependent Fermi sea. (The Fermi sea in S_+ which corresponds to a ground state in S_- will be described in section 4.)

3.3 Topology of AdS_2 and quantization of RR charge

In this subsection, we show that the topology of AdS_2 demands the RR flux to be quantized. This observation serves as a support to our identification of the matrix model S_+ with the global coordinates of AdS_2.

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The topology of AdS_2 can be defined by its embedding in $2+1$ dimensions

$$X_0^2 + X_{-1}^2 - X_1^2 = 1\,. \tag{3.6}$$

The Poincare patch is related to cartesian coordinates via the coordinate transformation

$$\sigma = (X_{-1} - X_1)^{-1}\,, \qquad t = \sigma X_0\,. \tag{3.7}$$

It has the metric

$$ds^2 = dX_0^2 + dX_{-1}^2 - dX_1^2 = \frac{1}{\sigma^2}(dt^2 - d\sigma^2)\,. \tag{3.8}$$

The global coordinates of AdS_2 is defined by (2.34), for which the metric is

$$ds^2 = \frac{1}{4\sin^2(\omega)}(d\tau^2 - d\omega^2)\,. \tag{3.9}$$

Translation in global time τ is generated by

$$L_0 = i\partial_\tau = H + K\,. \tag{3.10}$$

Hence AdS_2 in the Poincare patch is dual to S_0, and in global coordinates to S_+ [10].

According to the topology defined by (3.6), without extension to the covering space of AdS_2, τ is an angular variable, and so

$$e^{i2\pi L_0} = 1\,, \tag{3.11}$$

which implies that the eigenvalues of $L_0 = H + K$ have to be integers. L_0's eigenfunctions can be solved exactly [14] and the eigenvalues of are

$$E_n = n + |q|\,, \qquad n = 0, 1, 2, \dots\,, \tag{3.12}$$

where q is the RR charge of type 0A theory. The topological constraint (3.11) then requires that the RR charge q be quantized.

With the spectrum given by integers, the time variable t_+ of S_+ can be naturally compactified on the unit circle with $t_+ \in (-\pi, \pi)$. Recalling the image of t_0 in S_+ (2.32), we see that the spacetime of S_0 is half of the spacetime of S_+, in perfect agreement with the relation between the Poincare patch and global coordinates of AdS_2.

Since the matrix model for linear dilaton background is equivalent to adding a certain Fermi sea background in the AdS_2 matrix model, as we argued earlier, the quantization of q for AdS_2 ensures the quantization of q for other matrix models.

As a final remark on this issue, for $M > 0$, the classical description of S_+ has a period of π instead of 2π as shown by (2.23).[3] We need to examine the wave functions of S_+ in order to see that the period of t_+ is actually 2π. On the other hand, from the viewpoint of dual theories, it is a classical statement that the Poicare patch is half of the global AdS_2. This is hence an example of the fact that sometimes AdS/CFT duality matches classical effects to quantum effects.

[3] For $M = 0$, the classical period of t_+ is already 2π.

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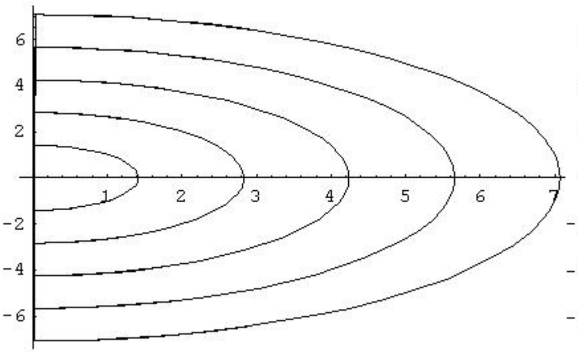

Figure 1: Phase space of S_+ for $M = 0$.

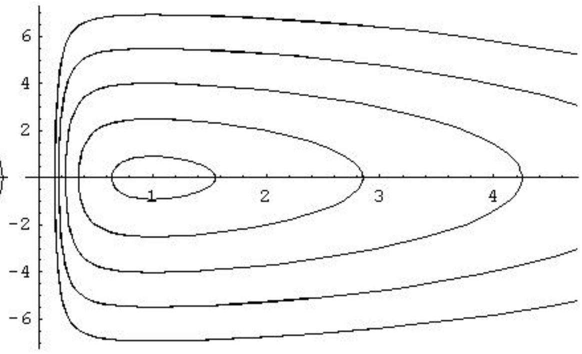

Figure 2: Phase space of S_+ for $M > 0$.

4. Fermi sea

For completeness, in this section we study how to transform a generic Fermi sea configuration in a certain coordinate system to another coordinate system.

The ground state of a matrix model is a Fermi sea filled to an energy μ. In another coordinate system this state is described as a time-varying Fermi sea, which can not be viewed as a small fluctuation over a static Fermi sea in view of the new hamiltonian. A configuration easy to discuss in one theory may be complicated for another.

A generic Fermi surface can be described by its boundary in the phase space $f(x, p, t) = 0$. As every point on the boundary has to follow the trajectories determined by the equation of motion (2.23)–(2.25) or (2.27)–(2.29), one can always rewrite $f(x, p, t)$ as a function of two variables. The two variables can be any two constants of motion, such as (E_0, T_0) in (2.24). Or instead we can use the coordinate and momentum at $t = 0$. Whichever variables we choose, we can use (2.21) and (2.22) to switch between different descriptions of a Fermi sea. The benefit of using the phase space variables at $t = 0$ is that they are identical in all theories.

For $M = 0$, the phase space evolution for S_+ is simply the circular periodic motion of the simple harmonic oscillator. (See figure 1.)[4] It is a periodic motion along deformed circles if $M > 0$. (See figure 2.) Classically the period for $M = 0$ is twice that for $M > 0$.

The phase space evolution for S_0 is straight horizontal motion along the x_0 axis if $M = 0$. (See figure 3.) For $M > 0$ a point in the phase space turns around at the infinite potential wall. The trajectories are asymptotically horizontal lines. (See figure 4.)

For S_-, a point in the phase space always moves along a hyperbolic curve. (See figure 5 and figure 6.)

It is then easy to visualize the Fermi surface in a different coordinate system. The phase space configuration of a Fermi surface at $t = 0$ in one coordinate system is identical in another coordinate system. As long as we know how each point in the phase space evolves with time, we can easily figure out the evolution of the Fermi surface in any theory.

[4]The lengths of curves are different because they are particle trajectories for the same period of time, i.e., faster particles leave a longer curve. For $M = 0$ the phase spaces have left-right symmetry and only the right half is plotted. For $M > 0$ only $x > 0$ is allowed.

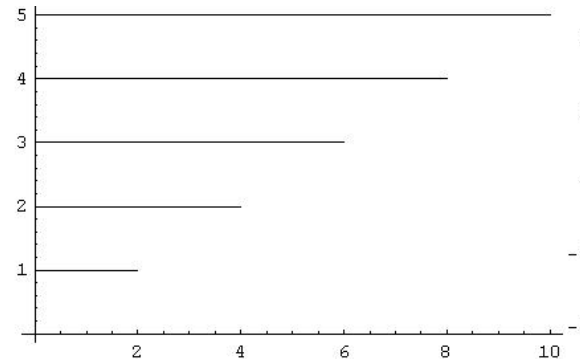

Figure 3: Phase space of S_0 for $M = 0$.

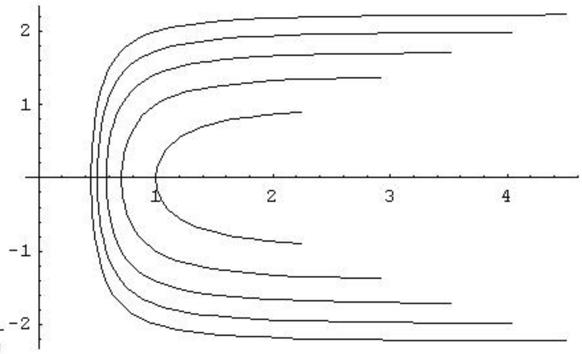

Figure 4: Phase space of S_0 for $M > 0$.

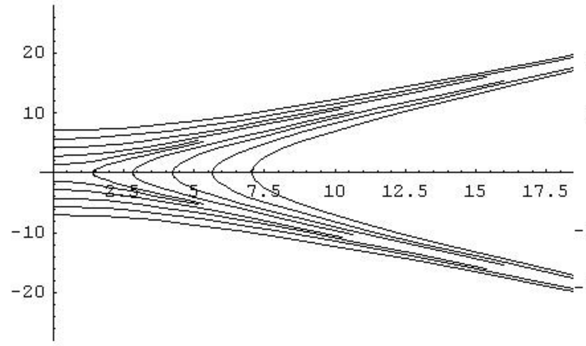

Figure 5: Phase space of S_- for $M = 0$.

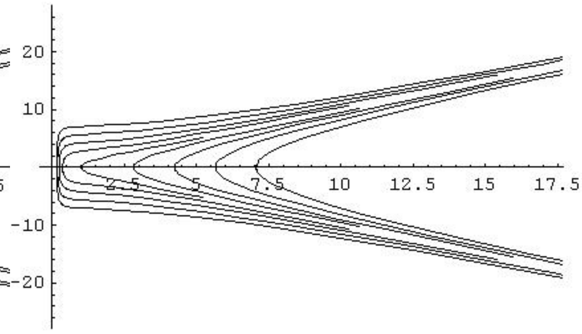

Figure 6: Phase space of S_- for $M > 0$.

For example, the static Fermi sea in the matrix model for bosonic strings is bounded by a hyperbolic curve in the phase space. At $t = 0$, it is static for S_-, and is rotating around the origin for S_+. In general a static Fermi sea is time-dependent in another coordinate system. The origin of the phase space for $M = 0$ is the only case that is static in all theories.

In [15], interesting time-dependent solutions of the Fermi sea were found in the matrix model. Some of them describe tachyon condensation. For instance, the solution for light-like tachyon condensation,

$$(x + p + 2\lambda e^t)(x - p) = g_s^{-1} \tag{4.1}$$

describes a Fermi sea which is at rest and filled up to the energy g_s^{-1} in the infinite past, but has all fermions removed to the infinities in the infinite future. However, in the S_+ theory all solutions are periodic. If tachyon condensation happens during half the cycle, the reverse process must take over the other half of the cycle. This is consistent with the fact that tachyons in 2 dimensions are not tachyonic.

5. Discussions

In this paper we clarified the meaning of the $\mathrm{SL}(2,\mathbb{R})$ symmetry as an isometry of the matrix model. We also used its Lie algebra generators, interpreted as Killing vectors of AdS_2 in the dual theory, to define new coordinate systems in which the matrix model takes different hamiltonians. This reflects the different appearances of AdS_2 in different coordinate patches. It turns out that all matrix models (1.7) are equivalent, and a choice of the string theory background corresponds to a choice of the Fermi sea. The moduli space of string theory is mapped to the space of all Fermi sea configurations (including the time-dependent ones).

As a supportive evidence of our interpretation, we used AdS_2's topological property to quantize the D0-brane charge q for the type 0A matrix model. We also find that the relation between the Poincare patch and global coordinates of AdS_2 is faithfully inherited by the map between time coordinates of corresponding matrix models.

Finally, let us examine more carefully to what extent we can claim the quantum equivalence among S_+, S_0, S_-. First, the Hilbert spaces, if defined as the spaces of normalizable wave functions, are identical even at the quantum level for all finite t (whenever the coordinate transformation is not singular), although the energy eigenfunctions are of course different. Yet if we further restrict ourselves to states with finite energies, the Hilbert spaces are different among S_+, S_0, S_-. For instance, according to (2.26),

$$E_+ = E_0(T_0+1)^2 + \frac{M}{4E_0}\,, \tag{5.1}$$

so E_+ diverges whenever $E_0 = 0$ or $T_0 \to \pm\infty$, which are finite energies in S_0. This is what one should expect as a generic phenomenon associated with spacetime coordinate transformations.

As we have seen from our discussion in section 2.3, the coordinate transformations are not bijective. We also have an example of inequivalence among S_+, S_0, S_-: the quantization of RR flux due to compactification of t_+ in S_+ can not be repeated in S_0 or S_- since the latter have no access to the full range of t_+. Similarly, we expect that S_+ will be superior to S_0 and S_- when we consider certain orbifolds of AdS_2, and that this conclusion may be carried over to the linear dilaton background when the appropriate Fermi sea is introduced.

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A. Solutions to the isometry condition

The isometry condition (2.4) can be solved for a given hamiltonian H. In fact it is possible to use the isometry condition to find all time-independent hamiltonians with nontrivial isometries.

Following section 2.1, if H has a quadratic term in ∂_x, and if V is time-independent, we can compare the coefficients of ∂_t on both sides of (2.4). It implies that

$$\mathcal{D}' = \mathcal{D} + i\frac{\partial}{\partial t}A\,. \tag{A.1}$$

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Then we compare the coefficient of $\partial_t\partial_x$, and see that we must have $\partial_x A = 0$, that is, A is a function of t only. hamiltonians with only linear terms in ∂_x will not be considered in this paper. But we expect that a similar approach will apply.

Matching the coefficients of ∂_x^n for each n in the isometry condition (2.4) gives

$$\partial_x B = \frac{1}{2}\partial_t A\,, \qquad \partial_x C = \partial_t B\,, \tag{A.2}$$

$$i\partial_t(C + AV) + iB\partial_x V + \frac{1}{2}\partial_x^2 C = 0\,. \tag{A.3}$$

The first two can be easily solved

$$B = \frac{1}{2}\partial_t A(t)x + b(t)\,, \qquad C = \frac{1}{4}\partial_t^2 A(t)x^2 + \partial_t b(t)x + c_0(t)\,, \tag{A.4}$$

and after plugging these in (A.3) we get

$$\partial_t A\left(V + \frac{1}{2}x\partial_x V\right) + b\partial_x V + \frac{1}{4}\partial_t^3 A x^2 + \partial_t^2 b x + \partial_t c_0 - \frac{i}{4}\partial_t^2 A = 0\,. \tag{A.5}$$

Now we should consider separately the cases $\partial_t A \neq 0$ and $\partial_t A = 0$.

If $\partial_t A \neq 0$, we can shift x by $x \to x - \frac{2b}{\partial_t A}$ to absorb b. Hence we can assume $b(t) = 0$ without loss of generality. We will still use x to stand for the shifted coordinate. The equations (A.4-A.5) are simplified to

$$B = \frac{1}{2}\partial_t A(t)x\,, \qquad C = \frac{1}{4}\partial_t^2 A(t)x^2 + c_0(t)\,, \tag{A.6}$$

$$\partial_t A(V + \frac{1}{2}x\partial_x V) + \frac{1}{4}\partial_t^3 A x^2 + \partial_t c_0 - \frac{i}{4}\partial_t^2 A = 0\,. \tag{A.7}$$

Eq. (A.7) is a statement about the linear dependence of $(V + \frac{x}{2}\partial_x V), x^2, x, 1$ as functions of x. Yet since the coefficients of them are functions of t, the only chance for it to hold for all t and x is that all coefficients are the same function of t up to overall constant factors. That is

$$\frac{\partial_t^3 A}{\partial_t A} = -8v_2\,, \qquad \frac{-\partial_t c_0 + \frac{i}{4}\partial_t^2 A}{\partial_t A} = v_0\,, \tag{A.8}$$

for some constants v_2, v_0. The potential V can then be easily solved from (A.7)

$$V = v_0 + v_2 x^2 + \frac{M}{2x^2}\,. \tag{A.9}$$

The isometry generators $\mathcal{D}$ are defined by A, B and C, which are determined by (A.6) and (A.8). For the potential (A.9) with $v_2 \neq 0$, the result is that $\mathcal{D}$ is in general a linear combination of the following 3 generators

$$\mathcal{H} = i\partial_t - v_0\,, \tag{A.10}$$

$$\mathcal{L}_+ = e^{\sqrt{-8v_2}t}\left(i\partial_t + \frac{i}{2}\sqrt{-8v_2}x\partial_x - 2v_2x^2 - \left(v_0 - \frac{i}{4}\sqrt{-8v_2}\right)\right), \tag{A.11}$$

$$\mathcal{L}_- = e^{-\sqrt{-8v_2}t}\left(i\partial_t - \frac{i}{2}\sqrt{-8v_2}x\partial_x - 2v_2x^2 - \left(v_0 + \frac{i}{4}\sqrt{-8v_2}\right)\right). \tag{A.12}$$

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They satisfy the SL$(2,\mathbb{R})$ Lie algebra

$$[\mathcal{H},\mathcal{L}_\pm] = \pm i\sqrt{-8v_2}\mathcal{L}_\pm\,, \qquad [\mathcal{L}_+,\mathcal{L}_-] = -2i\sqrt{-8v_2}\mathcal{H}\,. \tag{A.13}$$

For the potential (A.9) with $v_2 = 0$, i.e.,

$$V = v_0 + \frac{M}{2x^2}\,, \tag{A.14}$$

they are

$$\mathcal{H} = i\partial_t - v_0\,, \tag{A.15}$$

$$\mathcal{L}_1 = it\partial_t + \frac{i}{2}x\partial_x - v_0 t + \frac{i}{4}\,, \tag{A.16}$$

$$\mathcal{L}_2 = \frac{i}{2}t^2\partial_t + \frac{i}{2}tx\partial_x + \frac{x^2}{4} - \frac{v_0}{2}t^2 + \frac{i}{4}t\,. \tag{A.17}$$

They realize a different set of generators of the same algebra

$$[\mathcal{H},\mathcal{L}_1] = i\mathcal{H}\,, \qquad [\mathcal{H},\mathcal{L}_2] = i\mathcal{L}_1\,, \qquad [\mathcal{L}_1,\mathcal{L}_2] = i\mathcal{L}_2\,. \tag{A.18}$$

If on the other hand $\partial_t A = 0$, then A is a constant, and b has to be nonzero for the existence of any additional isometry generator in addition to $\mathcal{H}$. Due to $\partial_t A = 0$, (A.5) becomes

$$b\partial_x V + \partial_t^2 bx + \partial_t c = 0\,. \tag{A.19}$$

Again this implies that $b, \partial_t^2 b$ and $\partial_t c$ can only differ by constant factors. Thus V has to be of the form

$$V = v_2 x^2 + v_0\,, \tag{A.20}$$

and the symmetry generators are

$$\mathcal{M}_\pm = e^{\pm\sqrt{-2v_2}t}\left(i\partial_x \pm \sqrt{-2v_2}x\right), \tag{A.21}$$

in addition to $\mathcal{H} = i\partial_t$. $\mathcal{M}_\pm$ are space-like Killing operators. They satisfy the algebra

$$[\mathcal{H},\mathcal{M}_\pm] = \pm i\sqrt{-2v_2}\mathcal{M}_\pm\,, \qquad [\mathcal{M}_+,\mathcal{M}_-] = -2i\sqrt{-2v_2}\,. \tag{A.22}$$

The generators $\mathcal{L}_\pm$ (A.11, A.12) can still be defined when $v_{-2} = 0$. Their commutation relations with $\mathcal{M}_\pm$ are

$$[\mathcal{L}_\pm,\mathcal{M}_\pm] = 0\,, \qquad [\mathcal{L}_\pm,\mathcal{M}_\mp] = \mp 2i\sqrt{-2v_2}\mathcal{M}_\pm\,. \tag{A.23}$$

To summarize, for $M = 0$ in (1.7), there are a total of 5 isometry generators $\mathcal{H}, \mathcal{L}_\pm$ and $\mathcal{M}_\pm$. The two new generators $\mathcal{M}_\pm$ are in fact simply the creation and annihilation operators of the simple harmonic oscillator when $v_2 > 0$. (When $M \neq 0$, the SL$(2,\mathbb{R})$ algebra is also very useful for studying the spectrum [11, 14].) When $v_2 < 0$, they can be used to construct the discrete spectrum of imaginary energies in the matrix model.

If $v_2 = 0$, the new generators (A.21) reduce to

$$\mathcal{M}_1 = i\partial_x\,, \qquad \mathcal{M}_2 = it\partial_x + x\,. \tag{A.24}$$

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The algebra is defined by (A.18) and

$$[\mathcal{H}, \mathcal{M}_1] = 0\,, \qquad [\mathcal{H}, \mathcal{M}_2] = i\mathcal{M}_1\,, \qquad [\mathcal{M}_1, \mathcal{M}_2] = i\,, \tag{A.25}$$

$$[\mathcal{L}_1, \mathcal{M}_1] = -\frac{i}{2}\mathcal{M}_1\,, \qquad [\mathcal{L}_1, \mathcal{M}_2] = \frac{i}{2}\mathcal{M}_2\,, \tag{A.26}$$

$$[\mathcal{L}_2, \mathcal{M}_1] = -\frac{i}{2}\mathcal{M}_2\,, \qquad [\mathcal{L}_2, \mathcal{M}_2] = 0\,. \tag{A.27}$$

Using the fact that all hamiltonians with potentials of the form (A.9) with the same M are related to each other by coordinate transformations [11] (see next section), the generators (A.10)–(A.12) are related to the generators (A.15)–(A.17) by coordinate transformations. By unitary transformations of $e^{iv_0 t}$, we can always set $v_0 = 0$. Then we see that the generators in (A.15)–(A.17) are related to H, K, D in (1.5) as

$$(\mathcal{H}, \mathcal{L}_1, 2\mathcal{L}_2) \to (H, D, K)\,, \tag{A.28}$$

respectively, by setting $t = 0$ and replacing $i\partial_t$ by H.

B. Killing operator as hamiltonian

By suitably choosing a new time coordinate τ, we can rewrite the isometry generator $\mathcal{D}$

$$\mathcal{D} = \alpha\mathcal{H} + \beta\mathcal{L}_1 + 2\gamma\mathcal{L}_2 = if(t)\partial_t + \frac{i}{2}(\partial_t f(t))x\partial_x + \frac{i}{4}(\partial_t f(t) - i2\gamma x^2) \tag{B.1}$$

according to (A.15-A.17) as

$$i\partial_\tau\,, \tag{B.2}$$

where

$$f(t) = \alpha + \beta t + \gamma t^2 \tag{B.3}$$

for some constants α, β, γ, up to unitary transformations, so that τ is the coordinate translated by $\mathcal{D}$. Apparently, for the new Schrödinger equation

$$(i\partial_\tau - H_\tau)\psi = 0 \tag{B.4}$$

to be equivalent to the old one, H_τ should be equivalent to $\hat{\mathcal{D}}$, which is $\mathcal{D}$ with $i\partial_t$ replaced by H.

Comparing (B.1) with (B.2), we demand that

$$\frac{\partial t}{\partial \tau} = f(t)\,, \qquad \frac{\partial x}{\partial \tau} = \frac{1}{2}\partial_t f x\,. \tag{B.5}$$

They are solved by (2.10) and (2.12), or more explicitly

$$\tau = \tau(t) = \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(\frac{f'(t)}{\sqrt{\Delta}}\right), \qquad \sigma = f^{-1/2}x\,, \tag{B.6}$$

where $\Delta = 4\alpha\gamma - \beta^2$. Here σ is the new spatial coordinate. Obviously there is a freedom to shift τ by a constant.

For (B.6) to make sense we need to assume $\gamma \neq 0$. But they are still valid even if $\Delta < 0$, using $\tan(i\theta) = i\tanh(\theta)$. In terms of the new variables, the equation of motion $((\mathrm{EOM})) = i\partial_t - H$ becomes

$$(\mathrm{EOM}) = f^{-1}\left(i\partial_\tau + \frac{i}{4}(\partial_t f) + \frac{1}{2}(\partial_\sigma - \frac{i}{2}(\partial_t f)\sigma)^2 + \frac{1}{8}(\partial_t f)^2\sigma^2 - \frac{v_{-2}}{\sigma^2}\right) = f^{-1}(\mathrm{EOM})'. \tag{B.7}$$

One can find the new hamiltonian by reading it off from the new equation of motion. This expression can be simplified by a unitary transformation

$$\psi = U\hat{\psi}\,, \qquad U = e^{\frac{i}{4}(\partial_t f)\sigma^2}\,, \tag{B.8}$$

and a rescaling of the wave function

$$\hat{\psi} = f^{1/4}\tilde{\psi} \tag{B.9}$$

to absorb the time-dependent measure

$$\frac{dx}{d\sigma} = f^{-1/2}\,. \tag{B.10}$$

Let the new hamiltonian be defined through

$$i\partial_\tau - H_\tau = f^{1/4}U^\dagger\left(f(\mathrm{EOM})\right)Uf^{-1/4}\,, \tag{B.11}$$

We find

$$H_\tau = -\frac{1}{2}\partial_\sigma^2\frac{\Delta}{8}\sigma^2 + \frac{M}{2\sigma^2}\,. \tag{B.12}$$

Finally, one can check that

$$f^{1/4}U^\dagger \mathcal{D} U f^{-1/4} = i\partial_\tau\,, \tag{B.13}$$

as we aimed at in the beginning. Compared with section 2.2, H_τ (B.12) was denoted G in (2.13).

In the derivation above we see that, in addition to the coordinate transformation, a unitary transformation by U and a scaling by $f^{1/4}$ is needed for the wave function to satisfy Schrödinger equation in the new coordinate system.

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